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Research Statement

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My research focuses on problems in low-dimensional topology, specifically knot theory, contact geometry and symplectic topology. Knots and links are the basic objects to study in low-dimensional topology. For instance, any closed and oriented 3-manifold is the result of Dehn surgery along some knot or link in S^3. Researchers also study braids because any knot or link can be obtained from the closure of a braid. Furthermore, a braid is a mapping class where the surface is a punctured disk.

I am studying how to detect the conjugacy class of 3-braids admitting a flype by using Agol cycles [AK21]. I also like to classify knots and links. A knot invariant assigns a quantity to a knot and can be used to identify distinct knots when the output from the invariant of two knots differ. In [AKT21], I studied the s and τ knot concordance invariants and their application to contact topology. More details about my current and past research can be found in the subsequent sections.

Current Research

Let B_n denote the n stranded braid group. We focus on braids in B_3 which is isomorphic to the mapping class group of a 3-punctured disk, $\text{MCG}(D_3)$. In particular, we consider 3-braids that are related by a flype. The template of a negative flype from [BM06b] is shown in Figure 1.

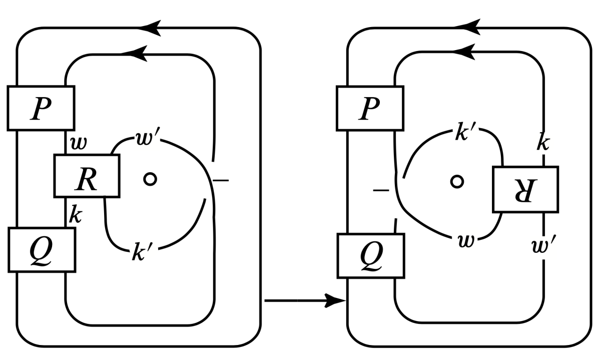

Figure 1. Negative flype template

Flypes are interesting to consider. For instance, in the Tait flype conjecture, Thistlethwaite and Menasco proved that two reduced alternating diagrams of an alternating link are related by a sequence of flypes [MT93]. In addition, flypes play a significant role in Birman and Menasco's work on Markov Theorem without Stabilization [BM06a] and on the existence of non transversally simple knots [BM06b]. Another specific result with flypes of 3-braids is Birman and Menasco's classification theorem in [BM93] which states that there exist two conjugacy classes of 3-braid representatives of link type $\mathcal{L}$ if and only if $\mathcal{L}$ has a 3-braid representative that admits a non-degenerate flype. Non-degenerate flypes have been classified by Ko and Lee in [KL99]. Although Birman and Menasco's paper was published in 1993, researchers have been unable to provide an alternate proof of such a fundamental result.

By restricting ourselves to pseudo-Anosov 3-braids, we are able to compare Agol cycles of braids related by a non-degenerate flype. First, we investigated the dilatation of a pseudo-Anosov mapping class, but it was ineffective in detecting a non-degenerate flype even though it is a conjugacy class invariant as detailed in Theorem 1.

Theorem 1. *Let $\beta = \sigma_1^x\sigma_2^{-1}\sigma_1^y\sigma_2^z$ and $\beta' = \sigma_1^x\sigma_2^z\sigma_1^y\sigma_2^{-1}$ be pseudo-Anosov 3-braids related by a flype. For large x, y and z, the braids β and β' have the same dilatation*

$$\lambda = \frac{1}{2}(\gamma + \sqrt{\gamma^2 - 4})$$

where $\gamma = sgn(xyz)(-2 - x - y + xz + yz + xyz)$. Thus the dilatation cannot detect non-degenerate flypes.

We consider Agol cycles which are a known conjugacy class invariant, specifically for pseudo-Anosov mapping classes [Ago11]. Agol cycles are sequences of invariant train tracks created from maximal splittings. The two train track types, Type M and W as shown in Figure 2, carry all train tracks of pseudo-Anosov 3-braids.

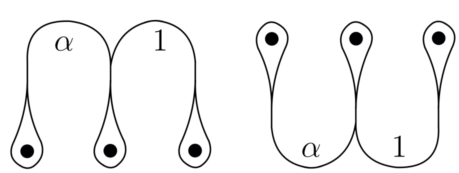

FIGURE 2. Train tracks M$(\alpha, 1)$ and W$(\alpha, 1)$

This research has led to define three types of Agol cycles: Agol-positive cycles which detect non-degenerate flypes, mirror-Agol positive cycles which detects non-degnerate flypes up to mirror image, and Agol-negative cycles which are unable to detect non-degenerate flypes. Our main result, Theorem 2, proves a sufficient condition of mirror Agol-positive. We also prove the existence of an infinite sequence of braids which satisfies the conditions of Theorem 2.

Theorem 2. *Let $\beta = \sigma_1^x \sigma_2^{-1} \sigma_1^y \sigma_2^z$ and $\beta' = \sigma_1^x \sigma_2^z \sigma_1^y \sigma_2^{-1}$ be pseudo-Anosov 3-braids related by a flype. Let α, α' be the edge weights of β and β' respectively as seen in Figure 2. Let $\{\tau_i\}_{i=1,2,\ldots}$ and $\{\tau'_{i'}\}_{i=1,2,\ldots}$ be maximal splitting sequences of train tracks for β and β' respectively. If*

(1) *α and α' satisfy $a + b\alpha + c\alpha' + d\alpha\alpha' = 0$ for some $a, b, c, d \in \mathbb{Z}$, and*
(2) *there exists a homeomorphism $\phi : D^2 \to D^2$ such that $\phi(\tau_l) = \texttt{mirror}(\tau'_m)$ for some $l, m \in \mathbb{N}$ that preserves the weights of the edges,*

then β and β' have the same Agol cycle up to scaling, homeomorphism and mirror image.

Past Research

The joint work I did with Keiko Kawamuro and Linh Truong in [AKT21] focused on comparing the values of Rasmussen's s invariant [Ras10] and Ozsváth and Szabó's τ invariant [OS03], the self-linking number of braids, and the four-ball genus of a knot. The slice-Bennequin inequality provides an upper bound for the self-linking number in terms of the four-ball genus and is one of the most significant inequalities in contact topology. We adopt the definition of the defect of the slice-Bennequin inequality from [HIK19] and similarly define the defects of the s-Bennequin and τ-Bennequin inequalities. In Theorem 3, we show that the defect from the slice-Bennequin inequality can be arbitrarily large while the defects from the s-Bennequin inequality and the τ-Bennequin inequality are both bounded. The knots from Theorem 3 are the closures of the braids seen in Figure 3 and are the first example of an infinite sequence of knots with this property.

Theorem 3. *There exists a family of knots K_n, where $n = 1, 2, \cdots$, such that the defect of the slice-Bennequin inequality is $2n$ and the defects of the s-Bennequin inequality and τ-Bennequin inequality are both zero.*

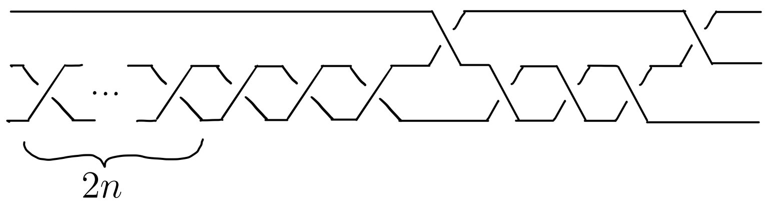

FIGURE 3. The braid β_n. The braid closure $K_1 = \widehat{\beta_1}$ is the knot 10_{125} and $K_2 = \widehat{\beta_2}$ is the knot $12n235$.

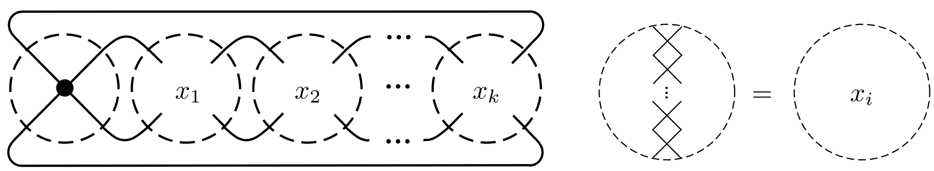

FIGURE 4. Pretzel projection $(x_1, x_2, \ldots, x_k)$

The knots from Theorem 3 also have applications to homological invariants. Ozsváth-Szabó-Thurston's transverse invariant $\hat{\theta}$ from knot Floer homology [OST08] and Plamevskaya's transverse invariant ψ from Khovanov homology [Pla06] both have the property of being nonzero for quasipositive knots. However, we show that the knots from Theorem 3 are non-quasipositive, $\hat{\theta}(K_n) \neq 0$, and $\psi(K_n) \neq 0$. Thus, these invariants are ineffective in detecting the non-quasipositive property for an infinite sequence of knots.

In [Ace17], I extended pseudodiagrams of knots and links along with their trivializing and knotting numbers to pseudodiagrams of spatial graphs. Pseudodiagrams were first introduced because precrossings (double points without over or under crossing information) appear naturally when observing DNA during the replication process. I explicitly calculated the trivializing and knotting numbers of 2-bouquet graphs based on the number of precrossings and their placement in the pretzel projection from Figure 4. In Theorem 4, I proved the existence of a pretzel projection with knotting number $k > 2$ and find the finite number of possible trivializing numbers.

Theorem 4. *For any $k \in \mathbb{N}$ where $k \geq 2$, there exists a pretzel projection $P = (x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n)$, where not both m and n are zero, $x_i = 1$ for all $1 \leq i \leq m$, $y_j \geq 2$ for all $1 \leq j \leq n$, with $kn(P) = k$, and the following trivializing numbers for some $l \in \mathbb{N}_0$:*

$$
tr(P) = \begin{cases} 2(k-1) \ \text{or}\ 2(k-2) & \text{if } n = 0 \\ 2n(k-2)+2+2l & \text{if } n \geq 1 \text{ and one } y_j \text{ is even} \\ 2\left\lfloor \frac{m+n}{2} \right\rfloor + 2n(k-2)+2l & \text{if } n \geq 1, \text{ all } y_j \text{ are odd,} \\ & \text{and } m \leq n+1 \\ 2(k-1)(n+1)+2l \ \text{or} & \text{if } n \geq 1, \text{ all } y_j \text{ are odd,} \\ 2\left\lfloor \frac{m+n}{2} \right\rfloor + 2n(k-2)+2l & \text{and } m > n+1 \end{cases}
$$

As an undergraduate, I worked with David Heywood, Ashley Klahr, and Oscar Vega in an REU in projective geometry and published our results in [AHKV14]. We developed a procedure to embed cycles of any length in $PG(n,q)$ given that q is a power of a prime and $n \geq 3$ as shown in Figure 5. This was the first time any results had been proven with $n \geq 3$. I also worked on a project as a Fresno State Tensor Women Scholar (supported by an MAA Tensor Women and Mathematics Grant) with Jennifer Elder. In [AE15], we constructed an invariant to distinguish between spatial graphs by transforming our graph into a knot and then using known invariants from knot theory to obtain an expression for each graph. By recording all possible link types that can be created from the original graph, we produced a stronger invariant. These processes can be manipulated so that they can be conducted on directed spatial graphs as well.

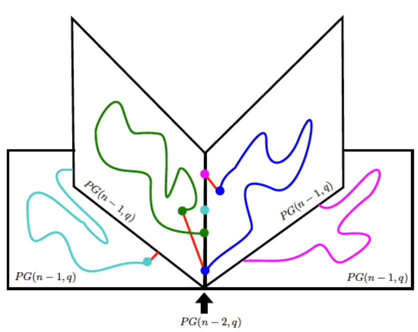

FIGURE 5. Visual representation of cycle construction in $PG(n,q)$

Future Research

During our investigation of Agol cycles of 3-braids, we noticed we could predict the length of the Agol cycle. If $\beta = \sigma_1^x\sigma_2^{-1}\sigma_1^y\sigma_2^z$ and $\beta' = \sigma_1^x\sigma_2^z\sigma_1^y\sigma_2^{-1}$ are related by a non-degenerate flype, their Agol cycles will have the same length and it is determined by the sign and values of x, y, and z as seen in Conjecture 1.

Conjecture 1. *If $\beta = \sigma_1^x\sigma_2^{-1}\sigma_1^y\sigma_2^z$ and $\beta' = \sigma_1^x\sigma_2^z\sigma_1^y\sigma_2^{-1}$ are related by a non-degenerate flype, then the length of the Agol cycle L is determined by the sign of x, y, and z and must be one of four values as outlined below.*

$$x,y,z<0 : L = |x|+|y|+|z|-5 \qquad y,z<0; x>0 : L = |x|+|y|+|z|-1$$

$$x,y<0; z>0 : L = |x|+|y|+|z|-3 \qquad y<0; x,z>0 : L = |x|+|y|+|z|-1$$

$$x,z<0; y>0 : L = |x|+|y|+|z|-1 \qquad z<0; x,y>0 : L = |x|+|y|+|z|+1$$

$$x<0; y,z>0 : L = |x|+|y|+|z|-1 \qquad x,y,z>0 : L = |x|+|y|+|z|-1$$

So far, we have only found Agol-positive and mirror Agol-positive cycles. However, we suspect that Agol-negative cycles exist which leads to the next conjecture.

Conjecture 2. *There exists an infinite sequence of 3-braids which admit a non-degenerate flype and have Agol-negative cycles.*

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